
AP[®] Calculus BC

Sample Student Responses and Scoring Commentary

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AP[®] CALCULUS BC
2019 SCORING GUIDELINES

Question 5

(a) $f'(x) = \frac{-(2x-2)}{(x^2-2x+k)^2}$

$$f'(0) = \frac{2}{k^2} = 6 \Rightarrow k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

3 : $\begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\frac{1}{x^2-2x-8} = \frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$

$$\Rightarrow 1 = A(x+2) + B(x-4)$$

$$\Rightarrow A = \frac{1}{6}, B = -\frac{1}{6}$$

3 : $\begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left(\frac{1}{6} \frac{1}{x-4} - \frac{1}{6} \frac{1}{x+2} \right) dx \\ &= \left[\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \right]_{x=0}^{x=1} \\ &= \left(\frac{1}{6} \ln 3 - \frac{1}{6} \ln 3 \right) - \left(\frac{1}{6} \ln 4 - \frac{1}{6} \ln 2 \right) = -\frac{1}{6} \ln 2 \end{aligned}$$

(c) $\int_0^2 \frac{1}{x^2-2x+1} dx = \int_0^2 \frac{1}{(x-1)^2} dx = \int_0^1 \frac{1}{(x-1)^2} dx + \int_1^2 \frac{1}{(x-1)^2} dx$

$$= \lim_{b \rightarrow 1^-} \int_0^b \frac{1}{(x-1)^2} dx + \lim_{b \rightarrow 1^+} \int_b^2 \frac{1}{(x-1)^2} dx$$

$$= \lim_{b \rightarrow 1^-} \left(-\frac{1}{x-1} \Big|_{x=0}^{x=b} \right) + \lim_{b \rightarrow 1^+} \left(-\frac{1}{x-1} \Big|_{x=b}^{x=2} \right)$$

$$= \lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \rightarrow 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\lim_{b \rightarrow 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

3 : $\begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

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NO CALCULATOR ALLOWED

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inf 2

5. Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$\begin{aligned}
 f(x) &= (x^2 - 2x + k)^{-1} \\
 f'(x) &= -(x^2 - 2x + k)^{-2} (2x - 2) \\
 &= -\frac{2x - 2}{(x^2 - 2x + k)^2} \\
 f'(0) &= -\frac{-2}{k^2} = 6 \\
 6k^2 &= 2 \\
 k &= \frac{1}{\sqrt{3}}
 \end{aligned}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x) dx$.

$$\begin{aligned}
 &\int_0^1 \frac{1}{x^2 - 2x + 8} dx \\
 &\int_0^1 \frac{1}{x-4} - \frac{1}{x+2} dx = \dots
 \end{aligned}$$

$$\frac{A}{x-4} + \frac{B}{x+2} = \frac{1}{(x-4)(x+2)}$$

$$A(x+2) + B(x-4) = 1$$

$$6A = 1 \quad A + B = 0$$

$$A = \frac{1}{6} \quad B = -\frac{1}{6}$$

$$\frac{1}{6} \ln|x-4| - \frac{1}{6} \ln|x+2| \Big|_0^1 =$$

$$\frac{1}{6} \ln \left| \frac{1-4}{1+2} \right| = \frac{1}{6} \ln \left| \frac{-3}{3} \right| - \frac{1}{6} \ln \left| \frac{-4}{2} \right| = \frac{1}{6} (0 - \ln 2) = \boxed{-\frac{1}{6} \ln 2}$$

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NO CALCULATOR ALLOWED

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2 of 2

(c) For $k = 1$, find the value of $\int_0^2 f(x) dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2 - 2x + 1} dx \quad \begin{array}{l} x^2 - 2x + 1 = 0 \\ (x-1)^2 = 0 \\ x = 1 \end{array}$$

$$\lim_{R \rightarrow 1^-} \int_0^R \frac{1}{(x-1)^2} dx + \lim_{R \rightarrow 1^+} \int_R^2 \frac{1}{(x-1)^2} dx$$

$$\lim_{R \rightarrow 1^-} -\frac{1}{x-1} \Big|_0^R + \lim_{R \rightarrow 1^+} -\frac{1}{x-1} \Big|_R^2$$

$$\lim_{R \rightarrow 1^-} \left(-\frac{1}{R-1} + \frac{1}{-1} \right) + \lim_{R \rightarrow 1^+} \left(-\frac{1}{2-1} + \frac{1}{R-1} \right)$$

$$\infty - 1 - 1 + \infty$$

The integral diverges

NO CALCULATOR ALLOWED

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172

5. Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2 - 2x + k)(0) - 1(2x - 2)}{(x^2 - 2x + k)^2}$$

$$f'(x) = \frac{-2x + 2}{(x^2 - 2x + k)^2} \quad f'(0) = 6$$

$$\frac{-2(0) + 2}{(0^2 - 2(0) + k)^2} = 6$$

$$\frac{2}{k^2} = 6$$

$$2 = 6k^2$$

$$k = \pm \sqrt{\frac{1}{3}}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x) dx$.

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$\int_0^1 \frac{1}{x^2 - 2x - 8} dx = \int_0^1 \frac{1}{(x-4)(x+2)} dx$$

$$= \frac{1}{6} \int_0^1 \frac{1}{x-4} dx - \frac{1}{6} \int_0^1 \frac{1}{x+2} dx$$

$$= \frac{1}{6} [\ln|x-4| \Big|_0^1 - \ln|x+2| \Big|_0^1]$$

$$= \frac{1}{6} [\ln 3 - \ln 4 - \ln 3 + \ln 2]$$

$$= \frac{1}{6} (\ln 2 - \ln 4)$$

$$= \frac{1}{6} \ln \frac{1}{2}$$

$$1 = A(x+2) + B(x-4)$$

$$1 = (A+B)x + 2A - 4B$$

$$(A+B=0) - 2$$

$$2A - 4B = 1$$

$$-2A - 2B = 0$$

$$-6B = 1$$

$$B = -\frac{1}{6}$$

$$A - \frac{1}{6} = 0$$

$$A = \frac{1}{6}$$

$$\frac{1}{(x-4)(x+2)} = \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}$$

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NO CALCULATOR ALLOWED

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272(c) For $k = 1$, find the value of $\int_0^2 f(x) dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2 - 2x + 1} dx$$

$$x^2 - 2x + 1 = (x-1)^2$$

$$\int_0^2 \frac{1}{(x-1)^2} dx \quad \text{let } u = x-1$$

$$\int_{-1}^1 \frac{1}{u^2} du$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$-\frac{1}{u} \Big|_{-1}^1$$

$$-\frac{1}{1} - \left(-\frac{1}{-1}\right)$$

$$\boxed{-2}$$

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NO CALCULATOR ALLOWED

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5. Consider the family of functions $f(x) = \frac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k , for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2 - 2x + k) - (2x - 2)}{(x^2 - 2x + k)^2} = \frac{x^2 - 4x + 2 + k}{(x^2 - 2x + k)^2}$$

$$0^2 - 4(0) + 2 + k = 6 \quad 2 + k = 6$$

$$k = 4$$

(b) For $k = -8$, find the value of $\int_0^1 f(x) dx$.

$$\int_0^1 \frac{1}{x^2 - 2x - 8} dx = \int_0^1 \frac{1}{(x-4)(x+2)} dx = \int_0^1 \frac{A}{x-4} + \frac{B}{x+2} dx = \frac{1}{6} \int_0^1 \frac{1}{x-4} dx - \frac{1}{6} \int_0^1 \frac{1}{x+2} dx$$

$$1 = Ax + 2A + Bx - 4B$$

$$A + B = 0 \quad 1 = -B$$

$$2A - 4B = 1$$

$$-6B = 1$$

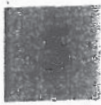
$$B = -\frac{1}{6} \quad A = \frac{1}{6}$$

$$\frac{1}{6} \left(\ln \frac{1}{3} - \ln \frac{1}{4} \right) - \frac{1}{6} \left(\ln \frac{1}{3} - \ln \frac{1}{2} \right)$$

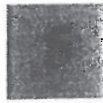
$$\frac{1}{6} \ln \frac{4}{3} - \frac{1}{6} \ln \frac{2}{3}$$

$$\frac{1}{6} \ln \frac{4}{2} = \frac{\ln 2}{6}$$

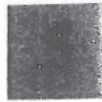
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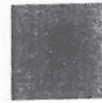
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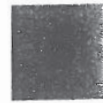
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NO CALCULATOR ALLOWED

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(c) For $k = 1$, find the value of $\int_0^2 f(x) dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2 - 2x + 1} dx = \int_0^2 \frac{1}{(x-1)^2} dx = \left. \frac{-1}{x-1} \right|_0^2 = \frac{-1}{1} - 1 = (-2)$$