2019

AP® CollegeBoard

AP® Calculus BC

Sample Student Responses and Scoring Commentary

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Question 5

(a) $f'(x) = \dfrac{-(2x-2)}{\left(x^2 - 2x + k\right)^2}$

$f'(0) = \dfrac{2}{k^2} = 6 \Rightarrow k^2 = \dfrac{1}{3} \Rightarrow k = \dfrac{1}{\sqrt{3}}$

$3 : \begin{cases} 1 : \text{denominator of } f'(x) \\ 1 : f'(x) \\ 1 : \text{answer} \end{cases}$

(b) $\dfrac{1}{x^2 - 2x - 8} = \dfrac{1}{(x-4)(x+2)} = \dfrac{A}{x-4} + \dfrac{B}{x+2}$

$\Rightarrow 1 = A(x+2) + B(x-4)$

$\Rightarrow A = \dfrac{1}{6},\ B = -\dfrac{1}{6}$

$$\int_0^1 f(x)\,dx = \int_0^1 \left(\frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2} \right) dx$$

$$= \left[\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2| \right]_{x=0}^{x=1}$$

$$= \left(\frac{1}{6}\ln 3 - \frac{1}{6}\ln 3 \right) - \left(\frac{1}{6}\ln 4 - \frac{1}{6}\ln 2 \right) = -\frac{1}{6}\ln 2$$

$3 : \begin{cases} 1 : \text{partial fraction decomposition} \\ 1 : \text{antiderivatives} \\ 1 : \text{answer} \end{cases}$

(c) $\displaystyle\int_0^2 \frac{1}{x^2 - 2x + 1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx$

$$= \lim_{b \to 1^-} \int_0^b \frac{1}{(x-1)^2}\,dx + \lim_{b \to 1^+} \int_b^2 \frac{1}{(x-1)^2}\,dx$$

$$= \lim_{b \to 1^-} \left(-\frac{1}{x-1} \Bigg|_{x=0}^{x=b} \right) + \lim_{b \to 1^+} \left(-\frac{1}{x-1} \Bigg|_{x=b}^{x=2} \right)$$

$$= \lim_{b \to 1^-} \left(-\frac{1}{b-1} - 1 \right) + \lim_{b \to 1^+} \left(-1 + \frac{1}{b-1} \right)$$

Because $\displaystyle\lim_{b \to 1^-} \left(-\frac{1}{b-1} \right)$ does not exist, the integral diverges.

$3 : \begin{cases} 1 : \text{improper integral} \\ 1 : \text{antiderivative} \\ 1 : \text{answer with reason} \end{cases}$

5A
1 of 2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$f(x) = (x^2-2x+k)^{-1}$

$f'(x) = -(x^2-2x+k)^{-2}(2x-2)$

$= -\dfrac{2x-2}{(x^2-2x+k)^2}$

$f'(0) = -\dfrac{-2}{k^2} = 6$

$6k^2 = 2$

$\boxed{k = \frac{1}{\sqrt{3}}}$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$\dfrac{A}{(x-4)} + \dfrac{B}{(x+2)} = \dfrac{1}{(x-4)(x+2)}$

$A(x+2) + B(x-4) = 1$

$6A = 1 \quad A + B = 0$

$A = \frac{1}{6} \quad B = -\frac{1}{6}$

$\int_0^1 \dfrac{1}{x^2-2x-8}\,dx$

$\int_0^1 \dfrac{\frac{1}{6}}{x-4} - \dfrac{\frac{1}{6}}{x+2}\,dx =$

$\frac{1}{6}\ln|x-4| - \frac{1}{6}\ln|x+2|\Big|_0^1 =$

$\frac{1}{6}\ln\left|\frac{x-4}{x+2}\right|\Big|_0^1 = \frac{1}{6}\ln\left|\frac{-3}{3}\right| - \frac{1}{6}\ln\left|\frac{-4}{2}\right| = \frac{1}{6}(0 - \ln 2) = \boxed{-\frac{1}{6}\ln 2}$

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(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2-2x+1}\,dx \qquad x^2-2x+1=0$$

$$(x-1)^2=0$$

$$x=1$$

$$\lim_{R\to 1^-}\int_0^R \frac{1}{(x-1)^2}\,dx + \lim_{R\to 1^+}\int_R^2 \frac{1}{(x-1)^2}\,dx$$

$$\lim_{R\to 1^-} -\frac{1}{x-1}\Big|_0^R + \lim_{R\to 1^+} -\frac{1}{x-1}\Big|_R^2$$

$$\lim_{R\to 1^-}\left(-\frac{1}{R-1}+\frac{1}{-1}\right) + \lim_{R\to 1^+}\left(-\frac{1}{2-1}+\frac{1}{R-1}\right)$$

$$\infty - 1 - 1 + \infty$$

$\boxed{\text{The integral diverges}}$

5B
1, 2, 2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2-2x+k)(0) - 1(2x-2)}{(x^2-2x+k)^2}$$

$$f'(x) = \frac{-2x+2}{(x^2-2x+k)^2}$$

$$f'(0) = 6$$

$$\frac{-2(0)+2}{(0^2-2(0)+k)^2} = 6$$

$$\frac{2}{k^2} = 6$$

$$2 = 6k^2$$

$$\boxed{k = \pm\sqrt{\frac{1}{3}}}$$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$$\frac{1}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x-4)$$

$$1 = (A+B)x + 2A - 4B$$

$$(A+B=0)-2$$

$$2A - 4B = 1$$

$$-2A - 2B = 0 \qquad A - \frac{1}{6} = 0$$

$$-6B = 1 \qquad A = \frac{1}{6}$$

$$B = -\frac{1}{6} \qquad \frac{1}{(x-4)(x+2)} = \frac{\frac{1}{6}}{x-4} - \frac{\frac{1}{6}}{x+2}$$

$$\int_0^1 \frac{1}{x^2-2x-8}\,dx = \int_0^1 \frac{1}{(x-4)(x+2)}$$

$$= \frac{1}{6}\int_0^1 \frac{1}{x-4}\,dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\,dx$$

$$= \frac{1}{6}\left[\ln|x-4|\Big|_0^1 - \ln|x+2|\Big|_0^1\right]$$

$$= \frac{1}{6}\left[\ln 3 - \ln 4 - \ln 3 + \ln 2\right]$$

$$= \frac{1}{6}(\ln 2 - \ln 4)$$

$$= \boxed{\frac{1}{6}\ln\frac{1}{2}}$$

Continue question 5 on page 19.

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$x^2 - 2x + 1 = (x-1)^2$

$\int_0^2 \frac{1}{x^2-2x+1}\,dx$

$\int_0^2 \frac{1}{(x-1)^2}\,dx$ let $u = x-1$

$\int_{-1}^1 \frac{1}{u^2}\,du$

$\frac{du}{dx} = 1$

$du = dx$

$-\frac{1}{u}\Big|_{-1}^{1}$

$-\frac{1}{1} - \frac{1}{1}$

$\boxed{-2}$

5C
1&2

5. Consider the family of functions $f(x) = \dfrac{1}{x^2 - 2x + k}$, where k is a constant.

(a) Find the value of k, for $k > 0$, such that the slope of the line tangent to the graph of f at $x = 0$ equals 6.

$$f'(x) = \frac{(x^2-2x+k)-(2x-2)}{(x^2-2x+k)^2} = \frac{x^2-4x+2+k}{(x^2-2x+k)^2}$$

$0^2 - 4(0) + 2 + k = 6 \quad 2+k=6$

$k=4$

(b) For $k = -8$, find the value of $\int_0^1 f(x)\,dx$.

$$\int_0^1 \frac{1}{x^2-2x-8}dx = \int_0^1 \frac{1}{(x-4)(x+2)}dx = \int_0^1 \frac{A}{x-4} + \frac{B}{x+2}\,dx = \frac{1}{6}\int_0^1 \frac{1}{x-4}\,dx - \frac{1}{6}\int_0^1 \frac{1}{x+2}\,dx$$

$1 = Ax + 2A + Bx - 4B$

$A + B = 0 \quad A = -B$

$2A - 4B = 1$

$-6B = 1$

$B = \frac{-1}{6} \quad A = \frac{1}{6}$

$\frac{1}{6}\left(\ln\frac{-1}{3} - \ln\frac{-1}{4}\right) - \frac{1}{6}\left(\ln\frac{1}{3} - \ln\frac{1}{2}\right)$

$\frac{1}{6}\ln\frac{4}{3} - \frac{1}{6}\ln\frac{2}{3}$

$\frac{1}{6}\ln\frac{4}{2} = \frac{\ln 2}{6}$

Continue question 5 on page 19.

NO CALCULATOR ALLOWED

5C 2 of 2

(c) For $k = 1$, find the value of $\int_0^2 f(x)\,dx$ or show that it diverges.

$$\int_0^2 \frac{1}{x^2-2x+1}\,dx = \int_0^2 \frac{1}{(x-1)^2}\,dx = \frac{-1}{x-1}\Big]_0^2 = \frac{-1}{1} - 1 = -2$$